NASA TT F-9455

HORIZONTAL AND AXISYMMETRIC TRANSITIONAL GAS DYNAMICS AT THE ANTERIOR BORDER OF A STREAMLINED BODY

NASA TT F-9455

K. A. Bezhanov

Translation of "O resheniyakh ploskikh i osesimmetrichnykh nestatsionarnykh zadach gazovoy dinamiki v okrestnosti peredney kromki obtekayemogo tela."
Izvestiya Akademii Nauk SSSR, Seriya Mekhanika, No. 2, pp. 168-170, 1965.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON DECEMBER 1965

HORIZONTAL AND AXISYMMETRIC TRANSITIONAL GAS DYNAMICS AT THE ANTERIOR BORDER OF A STREAMLINED BODY

- USSR -

[Following is a translation of an article by K. A. Bezhanov in the Russian-language periodical Izvestia AN SSSR, Mekhanika (News of the Academy of Sciences USSR, Mechanics), No 2, March-April 1962, pages 168-170.]

13480

Solutions are obtained for transitional equations of gas dynamics in horizontal and axisymmetric cases by expanding the desired functions into exponential series by the radius vector. For an arbitrary coefficient of the series we get a system of differential equations with variable coefficients, in which time acts as a parameter. The solution of this system is obtained in a closed form, containing arbitrary time functions which are determined by limit conditions on the shock wave and on the surface of the streamlined body. The obtained results allow calculation of gas flow in the region of the anterior border, or the cross-sectional view of the generatrix of the streamlined body in the case of parameters that do not change rapidly in time. In works [1 - 3] various specific cases of this report had been studied. In work [4] analogous series were used for studying free axisymmetric flows.

1. Statement of Problem

We will examine horizontal and axisymmetric gas flow in a moving system of coordinates, moving forward with speed V(t). In the horizontal case, we place the origin of coordinates at the anterior border of the body, and in the axisymmetric case -- at a certain distance R from the axis of symmetry, for example on the anterior border of a body with a channel or at the point of contour cross-section. We examine the flow in a coordinate system in which the position of a point is determined by radius vector r and angle θ, read from angle θ_s, where θ_s is the angle between the tangent to the anterior border and the axis of body symmetry.

The equation system of gas dynamics in the given set of coordinates has the form

$$
\begin{gathered}
r\frac{\partial u}{\partial t} + ru\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial \theta} - v^2 + \frac{r}{\rho}\frac{\partial p}{\partial r} = -r\frac{dV}{dt}\cos\theta_s \\
r\frac{\partial v}{\partial t} + ru\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial \theta} + uv + \frac{1}{\rho}\frac{\partial p}{\partial \theta} = r\frac{dV}{dt}\sin\theta_s \\
r\frac{\partial \rho}{\partial t} + \frac{\partial r\rho u}{\partial r} + \frac{\partial \rho v}{\partial \theta} + (\nu - 1)r\rho\frac{u\sin(\theta+\theta_s) + v\cos(\theta+\theta_s)}{R + r\sin(\theta+\theta_s)} = 0 \qquad (1.1) \\
r\frac{\partial}{\partial t}\frac{p}{\rho^\gamma} + ru\frac{\partial}{\partial r}\frac{p}{\rho^\gamma} + v\frac{\partial}{\partial \theta}\frac{p}{\rho^\gamma} = 0
\end{gathered}
$$

Here p -- pressure; ρ -- density; u and v -- projections of speed in the direction of the radius vector and perpendicular to it; γ -- adiabat indicator; $\nu = 1, 2$ correspondingly for horizontal and axisymmetric flows.

In solving system (1.1) the limiting conditions will be four conditions on the shock wave

$$
\begin{gathered}
\mathbf{V}_1\mathbf{s} = \mathbf{V}_2\mathbf{s}, \qquad \rho_1(\mathbf{V}_1 - \mathbf{D})\mathbf{n} = \rho_2(\mathbf{V}_2 - \mathbf{D})\mathbf{n} \\
\rho_1[(\mathbf{V}_1 - \mathbf{D})\mathbf{n}]^2 + p_1 = \rho_2[(\mathbf{V}_2 - \mathbf{D})\mathbf{n}]^2 + p_2 \qquad (1.2) \\
\frac{1}{2}[(\mathbf{V}_1 - \mathbf{D})\mathbf{n}]^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}[(\mathbf{V}_2 - \mathbf{D})\mathbf{n}]^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}
\end{gathered}
$$

and one condition on the body surface

$$
\mathbf{V}_2 \cdot \mathbf{n}^\circ = 0 \qquad (1.3)
$$

Indices 1 and 2 correspondingly indicate magnitudes related to the areas of flow before the shock wave and behind it; n, s, and n^o are unit vectors of the normal and tangent to the shock wave and the normal to the body surface; D is the relative speed of shock wave movement.

We seek the solution of system (1.1) in the form of the following exponential series:

$$u = \sum_{n=0}^{\infty} u_n(t,\theta) r^n, \qquad p = p_0(t) + \sum_{n=1}^{\infty} p_n(t,\theta) r^n$$
$$v = \sum_{n=0}^{\infty} v_n(t,\theta) r^n, \qquad \rho = \rho_0(t) + \sum_{n=1}^{\infty} \rho_n(t,\theta) r^n \tag{1.4}$$

where r indicates r/R.

2. Formation of Solution

We insert (1.4) into equations (1.1) and equate coefficients with the same powers r.

The equation for a zero approximation

$$\frac{\partial u_0}{\partial \theta} - v_0 = 0, \qquad u_0 + \frac{\partial v_0}{\partial \theta} = 0 \tag{2.1}$$

corresponds to quasistationary streamlining of a wedge with aperture semiangle equal to θ_S, and has the solution

$$u_0 = U(t)\cos\theta, \qquad v_0 = -U(t)\sin\theta \tag{2.2}$$

where U(t) is determined from the limit conditions. Introducing dimensionless variables

$$u_n/a_0, \quad v_n/a_0, \quad p_n/a_0^2\rho_0, \quad \rho_n/\rho_0, \quad a_0 = \sqrt{\gamma p_0/\rho_0}, \quad M = U/a_0$$

we get the following system for n expansion coefficients:

$$\begin{aligned}
&n\cos\theta u_n - \sin\theta\frac{\partial u_n}{\partial\theta} + \sin\theta v_n + \frac{n}{M}p_n = g_n(t,\theta)\\
&\sin\theta u_n - n\cos\theta v_n + \sin\theta\frac{\partial v_n}{\partial\theta} - \frac{1}{M}\frac{\partial p_n}{\partial\theta} = h_n(t,\theta)\\
&(n+1)u_n + \frac{\partial v_n}{\partial\theta} + nM\cos\theta\rho_n - M\sin\theta\frac{\partial\rho_n}{\partial\theta} = l_n(t,\theta)\\
&\sin\theta\frac{\partial}{\partial\theta}(\rho_n - p_n) - n\cos\theta(\rho_n - p_n) = q_n(t,\theta)
\end{aligned} \tag{2.3}$$

In system (2.3) time acts as a parameter, since the derivatives of the desired time functions are contained only in the right members. Integrating the fourth equation in (2.3), we get

$$\rho_n = p_n + A_n(t)\sin^n\theta + \sin^n\theta\int\frac{q_n d\theta}{\sin^{n+1}\theta} \tag{2.4}$$

where $A_n(t)$ is an arbitrary function of time.

We insert (2.4) into the third equation in (2.3) and eliminate u_n

$$\left(n^2 \cos\theta \operatorname{ctg}\theta + \frac{n}{\sin\theta} + \sin\theta\right) v_n - 2n\cos\theta \frac{\partial v_n}{\partial\theta} +$$
$$+ \sin\theta \frac{\partial^2 v_n}{\partial\theta^2} + \frac{n}{M} p_n + \frac{n+1}{M} \operatorname{ctg}\theta \frac{\partial p_n}{\partial\theta} - \frac{1}{M}\frac{\partial^2 p_n}{\partial\theta^2} = G_n(t,\theta) \qquad (2.5)$$
$$n(n+1)\operatorname{ctg}\theta v_n - n\frac{\partial v_n}{\partial\theta} + nM\cos\theta p_n - \left(M\sin\theta - \frac{n+1}{M\sin\theta}\right)\frac{\partial p_n}{\partial\theta} = H_n(t,\theta)$$

Substitution of the desired function in (2.5) by

$$v_n = w_n \sin^{n+1}\theta \qquad (2.6)$$

allows us to obtain, in the final result, a second order equation for p_n

$$2\cos\theta\sin^{n+1}\theta\frac{\partial w_n}{\partial\theta} + \sin^{n+2}\theta\frac{\partial^2 w}{\partial\theta^2} + \frac{n}{M}p_n + \frac{n+1}{M}\operatorname{ctg}\theta\frac{\partial p_n}{\partial\theta} - \frac{1}{M}\frac{\partial^2 p_n}{\partial\theta^2} = G_n(t,\theta)$$
$$n\sin^{n+1}\theta\frac{\partial w_n}{\partial\theta} - nM\cos\theta p_n + \left(M\sin\theta - \frac{n+1}{M\sin\theta}\right)\frac{\partial p_n}{\partial\theta} = H_n(t,\theta) \qquad (2.7)$$

From here, eliminating w_n, we get

$$(1 - M^2\sin^2\theta)\frac{\partial^2 p_n}{\partial\theta^2} + 2(n-1)M^2\sin\theta\cos\theta\frac{\partial p_n}{\partial\theta} +$$
$$+ n(n - M^2 - M^2(n-2)\cos^2\theta)p_n = Q_n(t,\theta) \qquad (2.8)$$

By the method of indeterminate coefficients we can find two linearly independent solutions of the corresponding homogeneous equation (2.8), since with substitution by $\sin^n\theta$ and $\cos^n\theta$ in this equation the result contains sines an cosines of power $< n$.

$$p_n = B_n(t)\sum_{k=0}^{n/2}\alpha_{2k}\sin^{2k}\theta + C_n(t)\sum_{k=0}^{n/2}\beta_{2k}\cos^{2k}\theta \qquad (2.9)$$

and with n not even

$$p_n = B_n(t)\sum_{k=1}^{(n+1)/2}\alpha_{2k-1}\sin^{2k-1}\theta + C_n(t)\sum_{k=1}^{(n+1)/2}\beta_{2k-1}\cos^{2k-1}\theta$$

where α_{2K}, β_{2K} and $\alpha_{2K-1}, \beta_{2K-1}$ are determined from a linear algebraic system, and $B_n(t)$ and $C_n(t)$ are arbitrary time functions. The specific solution of a non-homogeneous equation can be found by the method of indeterminate coefficients or by the method of constant variation. From (2.6) and the second equation of (2.7)

$$v_n = \frac{\sin^{n+1}\theta}{n} \int \frac{\partial\theta}{\sin^{n+1}\theta} \left(nM\cos\theta p_n - \left(M\sin\theta - \frac{n+1}{M\sin\theta} \right) \frac{\partial p_n}{\partial\theta} + H_n(t,\theta) + D_n(t) \right)$$

where $D_n(t)$ is an arbitrary function; u_n can be determined from the second equation of (2.3).

3. Limit Conditions

Let us express the shock wave equation and the body outline equation in a moving coordinate system in the form

$$\theta = \theta_0 + \theta_1(t)r + \theta_2(t)r^2 + \ldots, \quad \theta = \psi_0 + \psi_1(t)r + \psi_2(t)r^2 + \ldots \qquad (3.1)$$

For limit conditions in determining five unknown functions $A_n(t)$, $B_n(t)$, $C_n(t)$, $D_n(t)$ and $\theta_n(t)$, we have conditions (1.2) and (1.3). Let us substitute (3.1) in (1.2) and (1.3). For a zero approximation at $\theta = \theta_0$

$$u_{01} = u_{02}, \quad \rho_{01}v_{01} = \rho_{02}v_{02}, \quad \rho_{01}v_{01}^2 + p_{01} = \rho_{02}v_{02}^2 + p_{02} \qquad (3.2)$$
$$\frac{v_{01}^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_{01}}{\rho_{01}} = \frac{v_{02}^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_{02}}{\rho_{02}}, \quad u_{02} = U(t)\cos\theta, \quad v_{02} = -U(t)\sin\theta$$

The zero approximation corresponds to streamlining of a wedge, and the flow parameters for behind the shock wave at each moment of time are determined analogous to the stationary case. For an n approximation

$$u_{n1} + (n+1)v_{01}\theta_n = u_{n2} + (n+1)v_{02}\theta_n + f_n$$
$$\rho_{01}(v_{n1} - (n+1)u_{01}\theta_n) + \rho_{n1}v_{01} = v_{n2} - (n+1)u_{02}\theta_n + \rho_{n2}v_{02} + \lambda_n$$
$$2\rho_{01}v_{01}v_{n1} + \rho_{n1}v_{01}^2 + p_{n1} = 2v_{02}v_{n2} + \rho_{n2}v_{02}^2 + p_{n2} + \mu_n \qquad (3.3)$$
$$v_{01}(v_{n1} - (n+1)u_{01}\theta_n) + \gamma(\gamma-1)^{-1}(p_{n1}/\rho_{01} - p_{n1}\rho_{n1}/\rho_{01}^2)$$
$$= v_{02}(v_{n2} - (n+1)u_{02}\theta_n) + (\gamma-1)^{-1}(\gamma p_{n2} - \rho_{n2}) + \nu_n \quad \text{при } \theta = \theta_0 \qquad (3.3)$$
$$v_{n2} = (n+1)u_{02}\psi_n + \varkappa_n \quad \text{при } \theta = 0$$

[Legend]: a) at.

where the first index indicates the approximation number, and the second gives the area of flow, functions $f_n(t)$, $\lambda_n(t)$, $\mu_n(t)$, $\nu_n(t)$ and $\varkappa_n(t)$ depend on the preceding approximations. Substituting in (3.3) the solutions u_{n2}, v_{n2}, p_{n2} and ρ_{n2}, we get five linear algebraic equations for determining the five functions $A_n(t)$, $B_n(t)$, $C_n(t)$, $D_n(t)$ and $\theta_n(t)$.

As a simple example we calculated the streamlining of a conical body with a channel by a uniformly accelerated and a uniformly decelerated supersonic flow. At acceleration not more than $\pm$ 1000 meters/second^2, due to the smallness of parameter (R/a_{02}^2) (dU/dt) it is sufficient to view the flow at each moment of time as if the instantaneous flight speed were constant. An analogous fact holds in the case of potential streamlining of thin bodies [5, 6].

Submitted 14 April 1964

Bibliography

1. A. A. Dorodnitsyn, "The dependence of the curvature of the line of jump in compression on the curvature of the external surface of a revolving body with a channel," State Defense Publishers, Sb. teoreticheskikh rabot po aerodinamike (Symposium of theoretical works on aerodynamics), 1957.

2. G. G. Chernyy, Techeniye gaza c bol'shoy sverkhzvukovoy skorost'yu (Flow of gas with a great supersonic speed), Physico-mathematical Publishers, 1959.

3. U. G. Pirumov, K. A. Bezhanov, Ob odnom analiticheskom metode rascheta cverkhzvukovykh ploskikh i osesimmetricheskikh vikhrevykh techeniy gaza (On one analytical method for calculating supersonic horizontal and axisymmetric turbulent gas flow), State Defense Publishers, 1959.

4. V. P. Karlikov, "On the general theory of axisymmetric gas movement," DAN SSSR (Proceedings of the Academy of Sciences USSR), Vol 133, No 5, 1960.

5. F. I. Frankl', Ye. A. Karpovich, Gazodinamika tonkikh tel (Gas dynamics of thin bodies), State Technical Publishers, 1948.

6. D. U. Mayls, Potentsial'naya teoriya neustanovivshikhsya sverkhzvukovykh techeniy (Potential theory of irregular supersonic flow), Physico-mathematical Publishers, 1963.

- END -